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Unified inversion technique for fermion and boson integral equations

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A unified inversion technique for solving the fermion and boson integral equations is proposed. The method attributes the inversion of a convolution integral equation to that of a Toeplitz matrix. It may represent a general approach for treating a convolution integral equation.

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The inverse problems for boson and fermion systems have for a long period been of importance and attracted much research interest. However, there existed few exact analytical expressions for these problems before. It was not until recently that Chen put forward two independent methods and gave a very concise analytical solution for each problem [1–6]. One of Chen's methods applies the well-known Möbius inversion transform in arithmetic number theory [1,2], the other is related to a new expression for the Dirac δ function [5–7]. In this paper, we propose an inversion method, that is able to reproduce the results of Chen's two unrelated methods. The unified method, which we name *the matrix inversion method*, can be regarded as a universal trick for solving the convolution integral equations that frequently appear in physics and technology. As a further example, we discuss its application to a singular integral equation, the Abel equation, which has been extensively investigated in mathematics and mechanics [8].

I. MATRIX INVERSION METHOD

Many problems in physical and technological sciences are found to be related to such a convolution integral equation as

$$P(x) = \int_{-\infty}^{+\infty} Q(y)\Phi(y-x)dy. \tag{1}$$

As usual, Eq. (1) can in principle be inverted by using the deconvolution method. However, the deconvolution method not only reveals little but fails in some cases, because it is difficult to perform the Fourier transform for some functions. This is the reason that motivates us to develop a different method. Our starting point is to expand $Q(y)$ into a Taylor series around x so as to rewrite Eq. (1) as

$$P(x) = \sum_{n=0}^{\infty} A_n Q^{(n)}(x), \tag{2}$$

where

$$A_n = \frac{1}{n!}\int_{-\infty}^{+\infty} t^n \Phi(t)dt. \tag{3}$$

By derivation with respect to x, step by step, we have

$$P^{(1)}(x) = \sum_{n=0}^{\infty} A_n Q^{(n+1)}(x),$$

$$P^{(2)}(x) = \sum_{n=0}^{\infty} A_n Q^{(n+2)}(x),$$

$$\vdots$$

$$P^{(k)}(x) = \sum_{n=0}^{\infty} A_n Q^{(n+k)}(x),$$

$$\vdots$$

The above equations can be written into a matrix formula

$$\begin{pmatrix} P^{(0)}(x) \\ P^{(1)}(x) \\ \vdots \\ P^{(k)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_k & \cdots \\ 0 & A_0 & A_1 & \cdots & A_{k-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & A_0 & \\ & & & & \vdots & \end{pmatrix} \begin{pmatrix} Q^{(0)}(x) \\ Q^{(1)}(x) \\ \vdots \\ Q^{(k)}(x) \\ \vdots \end{pmatrix}. \tag{4}$$

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The inversion of Eq.(4) straightforwardly reads

$$\begin{pmatrix} Q^{(0)}(x) \\ Q^{(1)}(x) \\ \vdots \\ Q^{(k)}(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_k & \cdots \\ 0 & B_0 & B_1 & \cdots & B_{k-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & B_0 & \\ & & & & \vdots & \end{pmatrix} \begin{pmatrix} P^{(0)}(x) \\ P^{(1)}(x) \\ \vdots \\ P^{(k)}(x) \\ \vdots \end{pmatrix}. \tag{5}$$

The matrix **B** is the inverse of **A**: $\mathbf{B}=\mathbf{A}^{-1}$. Both **A** and **B** are the so-called Toeplitz matrices. Then $Q(x)$ can be expressed as

$$Q(x) = \sum_{n=0}^{\infty} B_n P^{(n)}(x), \tag{6}$$

where the coefficient B_n can be determined by a discrete convolution relation

$$\sum_{k=0}^{n} A_k B_{n-k} = \delta_{n0}, \tag{7}$$

where δ_{n0} is the Kronecker delta. Thus we have shown the basic idea of *the matrix inversion method.* The method establishes a relation between two function spaces, one spanned by the derivatives $\{P^{(n)}(x), n = 0,1,2,\ldots\}$ and the other by $\{Q^{(n)}(x), n = 0,1,2,\ldots\}$. The relation is expressed as a Toeplitz matrix. The inversion of the integral equation is then attributed to the inversion of the Toeplitz matrix. This leads to a discrete convolution equation (7) from which the inversion coefficients can be determined. In general, the approach will be applicable under the condition that the integral of Eq. (3) exist for any n. Evidently, the requirement of this condition upon the kernel function is $\lim_{t\to\pm\infty}\Phi(t) = 0$, with a convergent speed no slower than an exponential function.

From Eq. (7) we introduce a useful equality between the characteristic functions of the two progressions $\{A_n\}$ and $\{B_n\}$,

$$\sum_{n=0}^{\infty} A_n z^n = 1 \Big/ \left[\sum_{n=0}^{\infty} B_n z^n\right]. \tag{8}$$

This equality can be used to determine the inversion coefficients B_n.

II. DERIVATION OF CHEN'S FORMULA FOR FERMION INTEGRAL EQUATION

The so-called fermion integral equations refer to those whose kernels look like or can be converted to a Fermi distribution function,

$$P(x) = \int_{-\infty}^{+\infty} \frac{Q(y)}{1+\exp(y-x)} dy. \tag{9}$$

Chen *et al.* have indicated that a few problems such as the inversion of the relaxation-time distribution from the dielectric function spectra and the inverse isotherm problems for the adsorption energy distribution can be ascribed to a fermion integral equation [6]. The first-order derivative with respect to x of the left-hand side and the right-hand side of Eq. (9) is

$$P^{(1)}(x) = \int_{-\infty}^{+\infty} \frac{\exp(y-x)}{[1+\exp(y-x)]^2} Q(y) dy. \tag{10}$$

According to the matrix inversion method, Eq. (10) can be written as

$$P^{(1)}(x) = \sum_{n=0}^{\infty} A_n Q^{(n)}(x). \tag{11}$$

The solution of Eq. (11) is namely

$$Q(x) = \sum_{n=0}^{\infty} B_n P^{(n+1)}(x). \tag{12}$$

Let us see how to use Eq. (8) to find B_n. First it can be shown that

$$\sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{(zt)^n e^t}{(1+e^t)^2} dt = \int_0^{\infty} \frac{t^z}{(1+t)^2} dt$$
$$= B(1-z, 1+z) = \frac{\Gamma(1-z)\Gamma(1+z)}{\Gamma(2)} = \frac{z\Gamma(1-z)\Gamma(z)}{\Gamma(2)} = \frac{\pi z}{\sin(\pi z)}, \tag{13}$$

where $B(x)$ is the B function and $\Gamma(x)$ the Γ function. According to Eq. (8) we have

$$\sum_{n=0}^{\infty} B_n z^n = \frac{\sin(\pi z)}{\pi z}. \tag{14}$$

Therefore the inversion coefficients B_n are

$$B_n = \begin{cases} 0, & n = 2m+1, \\ (-1)^m \pi^{2m}/(2m+1)!, & n = 2m. \end{cases} \tag{15}$$

From Eqs. (12) and (15), the inversion formula for the fermion system can be seen [6]:

$$\begin{aligned} Q(x) &= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}\frac{\partial^{2n+1}}{\partial x^{2n+1}}P(x) \\ &= \frac{1}{\pi}\sin\left(i\pi\frac{\partial}{\partial x}\right)P(x) \\ &= \frac{1}{2i\pi}[P(x+i\pi)-P(x-i\pi)]. \end{aligned} \tag{16}$$

The method for deriving the above formula in Ref. [6] stands for an idea of representing the Dirac δ function by a kernel with a special operator. Hence it is related to the inverse problem of the Green function method for the differential equation [7].

III. DERIVATION OF THE CHEN-MÖBIUS FORMULA FOR THE BOSON INTEGRAL EQUATION

The boson integral equations stand for a sort of integral equation representing some physics of boson systems, such as phonons, photons, spin waves, and so on. Traced to Chen's pioneering work, the inversion problems of the boson system have involved a variety of problems, such as the analysis of the dust temperature distributions in star-forming condensations [9], the determination of the temperature distribution of the material shells of distorted black holes from their Hawking signals [10], the inverse blackbody radiation problem [11,12], and so on. A simple well-known problem is the inversion of the phonon density of states from experimental specific heat as a function of temperature, which was solved by Chen with the Möbius inversion transform [1]

$$\frac{\theta^2}{rk}C_v(\hbar\theta/k) = \int_0^{\infty}\frac{e^{\omega/\theta}}{(e^{\omega/\theta}-1)^2}\omega^2 g(\omega)d\omega, \tag{17}$$

where k is the Boltzmann constant, $\hbar$ is the Planck constant, r is the number of atoms in a unit cell, and $\theta = kT/\hbar$. Now let us see how to solve Eq. (17) with the use of the present method. We rewrite Eq. (17) by letting $e^y = \omega$, $e^x = \theta$,

$$\frac{e^{2x}}{rk}C_v(\hbar e^x/k) = \int_{-\infty}^{+\infty}\frac{\exp(e^{y-x})}{[\exp(e^{y-x})-1]^2}e^{3y}g(e^y)dy. \tag{18}$$

In order to get a suitable kernel, we rewrite Eq. (18) as

$$\frac{1}{rk}e^{(2-\lambda)x}C_v(\hbar e^x/k) = \int_{-\infty}^{+\infty}\frac{\exp(e^{y-x})e^{\lambda(y-x)}}{[\exp(e^{y-x})-1]^2}e^{(3-\lambda)y}g(e^y)dy. \tag{19}$$

By this means we reach a convolution-type equation, with $P(x) = (1/rk)e^{(2-\lambda)x}C_v(\hbar e^x/k)$ and $Q(x) = e^{(3-\lambda)x}g(e^x)$. The parameter $\lambda \geq 3$ is introduced to guarantee the existence of the integral,

$$A_n = \frac{1}{n!}\int_{-\infty}^{+\infty}\frac{t^n\exp(e^t+\lambda t)}{[\exp(e^t)-1]^2}dt. \tag{20}$$

The remaining question is how to decide the inversion coefficients B_n. Similar to the treatment in the above section, we have

$$\begin{aligned} \sum_{n=0}^{\infty}A_n z^n &= \int_{-\infty}^{+\infty}\frac{e^{zt}\exp[e^t+\lambda t]}{[\exp(e^t)-1]^2}dt \\ &= \int_0^{\infty}\frac{e^t t^{z+\lambda-1}}{[e^t-1]^2}dt = \int_0^{\infty}\frac{e^{-t}t^{z+\lambda-1}}{[1-e^{-t}]^2}dt \\ &= \sum_{n=0}^{\infty}\int_0^{\infty}t^{\lambda+z-1}ne^{-nt}dt \\ &= \Gamma(\lambda+z)\zeta(\lambda+z-1), \end{aligned} \tag{21}$$

where $\zeta(x)$ is the Riemann ζ function. In order to derive the Chen-Möbius formula [Eq. (15) in Ref.[1]], let us first show what the inverse Laplace operator will be within the framework of our theory. A Laplace transform, such as

$$\overline{\psi}(p/n) = \int_0^{\infty}e^{-pt/n}\psi(t)dt, \tag{22}$$

can be converted into Eq. (1), with $P^L(x) = \overline{\psi}(e^{-x}/n)e^{-\lambda x}$, $Q^L(y) = \psi(e^y)e^{(1-\lambda)y}$, and $\Phi^L(u) = e^{\lambda u}\exp(-e^u/n)$. Then

$$\sum_{n=0}^{\infty}B_m^L(n)z^m = [n^{\lambda+z}\Gamma(\lambda+z)]^{-1} \tag{23}$$

and

$$\psi(x) = \sum_{m=0}^{\infty}x^{\lambda-1}B_m^L(n)\left(x\frac{d}{dx}\right)^m[x^{-\lambda}\overline{\psi}(1/nx)] \tag{24}$$

or

$$\psi(x/n) = n\sum_{m=0}^{\infty}x^{\lambda-1}B_m^L(n)\left(x\frac{d}{dx}\right)^m[x^{-\lambda}\overline{\psi}(1/x)], \tag{25}$$

where $B_m^L(n)$ is the inversion coefficient for the inverse Laplace transform. Note that we have used $d^n/dx^n = (e^x d/de^x)^n$. According to Chen's definition, the inverse Laplace operator L_n^{-1} inverts the u space to ω/n space ($u = 1/\theta$ is the coldness): $\psi(x/n) = L_n^{-1}\overline{\psi}(1/x)$. Therefore, Eq. (25) is just the expansion of L_n^{-1} into the combination of the operator xd/dx. Now we have

$$\begin{aligned} \sum_{m=0}^{\infty}B_m z^m &= \frac{1}{\Gamma(\lambda+z)\zeta(\lambda+z-1)} \\ &= \sum_{n=1}^{\infty}\mu(n)n\frac{1}{\Gamma(\lambda+z)n^{\lambda+z}} \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\mu(n)nB_m^L(n)z^m, \end{aligned}$$

i.e.,

$$B_m = \sum_{n=1}^{\infty} \mu(n) n B_m^L(n), \tag{26}$$

where $\mu(n)$ is the arithmetical Möbius function. For Eq. (26) we have used the relation $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$. Our method gives the inversion result as

$$x^2 g(x) = \frac{1}{rk} \sum_{m=0}^{\infty} x^{\lambda-1} B_m \left(x\frac{d}{dx}\right)^m \left[x^{-\lambda} x^2 C_v \left(\frac{\hbar x}{k}\right)\right]. \tag{27}$$

According to Eq. (26), it becomes

$$x^2 g(x) = \frac{1}{rk} \sum_{n=1}^{\infty} \mu(n) n \sum_{m=0}^{\infty} x^{\lambda-1} B_m^L(n) \left(x\frac{d}{dx}\right)^m \times \left[x^{-\lambda} x^2 C_v \left(\frac{\hbar x}{k}\right)\right]. \tag{28}$$

Noting Eq. (25), one can see that Eq. (28) is just the general Chen-Möbius formula

$$g(\omega) = \frac{1}{rk\omega^2} \sum_{n=1}^{\infty} \mu(n) L_n^{-1} \left[\frac{1}{u^2} C_v \left(\frac{\hbar}{ku}\right)\right]. \tag{29}$$

IV. DISCUSSION AND CONCLUSIONS

In the foregoing text, we have demonstrated our method and its applications for solving two important integral equations in statistical physics. It is shown that the fermion and boson integral equations can be treated within an identical framework. Of more importance, the unified method represents a general and powerful approach to more than the enumerated examples. First of all, the method provides a possible numerical treatment for convolution-type integral equations (this possibility will be discussed in another paper). Second, when it is used to seek an analytical solution, one can avoid the analytical extension of the equation to the whole complex plane, which is perhaps inevitable in other methods [6,13]. Finally, it can be used to solve other kinds of integral equations. For example, it may be worthwhile to discuss the application to the well-studied Abel integral equation,

$$G(s) = \int_0^s \frac{F(t)}{(s-t)^\alpha} dt = \int_0^\infty \frac{\theta(s-t)}{(s-t)^\alpha} F(t) dt \quad (0 < \alpha < 1). \tag{30}$$

By letting $s = e^x$ and $t = e^y$, it can be translated into Eq. (1), with $P(x) = e^{\alpha x} G(e^x)$, $Q(y) = e^y F(e^y)$, and $\Phi(t) = [1-\theta(t)]/(1-e^t)^\alpha$. Therefore,

$$\sum_{n=0}^{\infty} A_n z^n = \int_{-\infty}^{+\infty} \frac{e^{zt}[1-\theta(t)]}{(1-e^t)^\alpha} dt = \int_0^1 t^{z-1}(1-t)^{-\alpha} dt = B(z, 1-\alpha), \tag{31}$$

or

$$\sum_{n=0}^{\infty} B_n z^n = \frac{\Gamma(z+1-\alpha)}{\Gamma(z)\Gamma(1-\alpha)}. \tag{32}$$

The inverse solution is

$$e^x F(e^x) = \sum_{n=0}^{\infty} B_n \frac{d^n}{dx^n} [e^{\alpha x} G(e^x)]. \tag{33}$$

The solution will be especially easy when $G(s)$ is a polynomial. Let us consider the simplest case, $G(s) = C$; then the result will be

$$F(x) = C \sum_{n=0}^{\infty} B_n \alpha^n x^{\alpha-1} = \frac{C\Gamma(1)}{\Gamma(1-\alpha)\Gamma(\alpha)} x^{\alpha-1} = \frac{C}{\pi} \sin(\pi\alpha) x^{\alpha-1}. \tag{34}$$

A famous example (a tautochrone) is the following: a particle slides along a smooth curve $z(x)$ in the vertical plane with zero initial velocity; if the time it takes the particle to move from $z = h$ to $z = 0$ is a constant T, then we find the trajectory $z(x)$. According to classical mechanics, this problem can be formulated as

$$T = \int_0^h \sqrt{\frac{1 + (dx/dz)^2}{2g(h-z)}} dz. \tag{35}$$

In this case, $\alpha = 1/2$ and $F(z) = [1 + (dx/dz)^2]^{1/2}$. According to Eq. (34), we have

$$1 + \left(\frac{dx}{dz}\right)^2 = \frac{2gT^2}{\pi^2 z}. \tag{36}$$

This is exactly the cycloid equation.

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